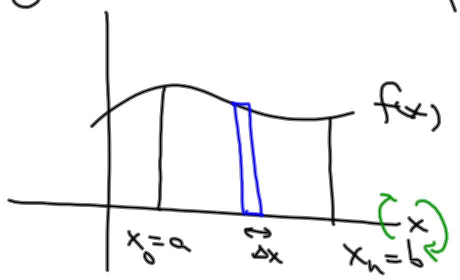
7.1 - 7.3 Volume:

(1) Method: Disk or Washer

When you rotate a region between two curves, about an axis, the segments (or rectangular representatives) that are perpendicular to the axis of revolution produce disks which integral give the volume of the solid of revolution

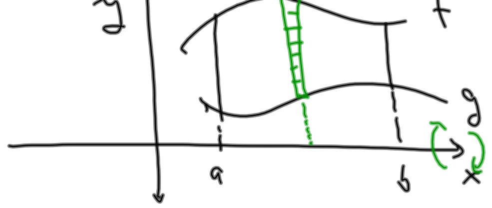


$$\lim_{n \to \infty} \sum_{i=0}^{n} A(x_i)\Delta x \quad \text{where } A(x_i) = \text{area of disks!}$$

$$V = \pi \int_a^b (\text{radius})^2 dx = \pi \int_a^b [f(x)]^2 dx$$

Disk method!

In general,



$$V = \pi \int_a^b f^2 dx - \pi \int_a^b g^2 dx$$

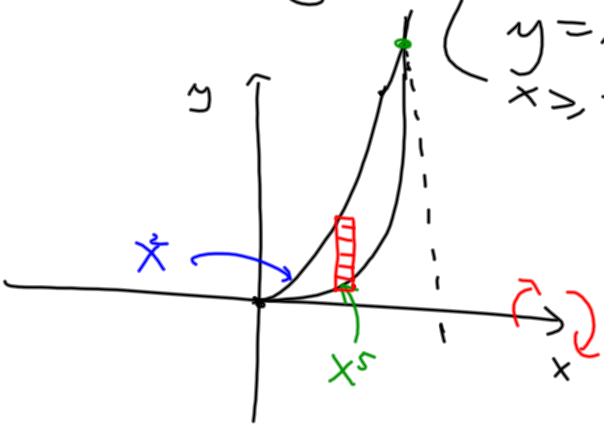
$$= \pi \int_a^b (f^2 - g^2)\, dx$$

Washer method!

Ex:

Region: $\begin{cases} y = x^2 \\ y = x^5 \\ x \geq 0 \end{cases}$

Rotation about x-axis



$$V = \pi \int^{1} \left(\text{outer radius}\right)^2 - \left(\text{inner radius}\right)^2 dx$$

$$= \pi \int_0^1 \left([x^2]^2 - [x^5]^2 \right) dx$$

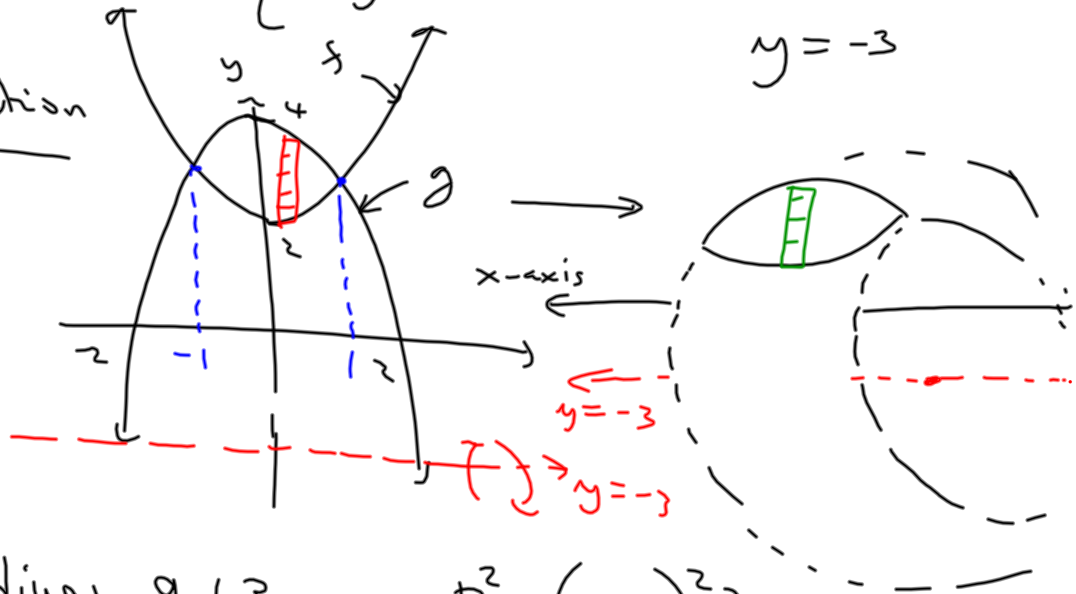
$$= \pi \left[\frac{1}{5}x^5 - \frac{1}{11}x^{11} \right]_0^1$$

$$V = \pi \left[\frac{1}{5} - \frac{1}{11} \right] = \frac{6\pi}{55} \text{ unit}^3$$

Ex :

Region $\begin{cases} f(x) = x^2 + 2 \\ g(x) = 4 - x^2 \end{cases}$ Rotate about $y = -3$

Solution



Outer radius: $g + 3 \longrightarrow R^2 = (g+3)^2$

inner radius: $f + 3 \longrightarrow r^2 = (f+3)^2$

$\Bigg\} \longrightarrow V = \pi \int_{-1}^{1} (R^2 - r^2)\,dx$

$$\text{Volume} = \pi \int_{-1}^{1} \left[(\underbrace{4 - x^2}_{g} + 3)^2 - (\underbrace{(x^2+2)}_{f} + 3)^2 \right] dx$$

$$= \pi \int_{-1}^{1} \left[(7 - x^2)^2 - (x^2 + 5)^2 \right] dx$$

$$= \pi \int_{-1}^{1} (49 - 14x^2 + x^4 - x^4 - 10x^2 - 25)\,dx = \pi \int_{-1}^{1} (-24x^2 + 24)\,dx$$

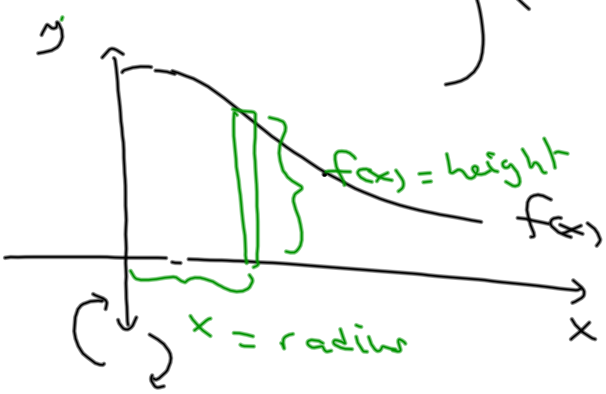
$$= \pi \left[-8x^3 + 24x \right]_{-1}^{1} = \pi \left[-8 + 24 - 8 + 24 \right]$$

$$\boxed{V = 32\pi \text{ unit}^3}$$

② Shell method

If we rotate about an axis a region between two curves, the segments (or rectangular representatives) that are parallel to the axis of revolution produce cylindrical shells. The integral of the areas of these shells give the volume of the solid of revolution:

$$V = 2\pi \int (\text{radius})(\text{height})\, dx$$



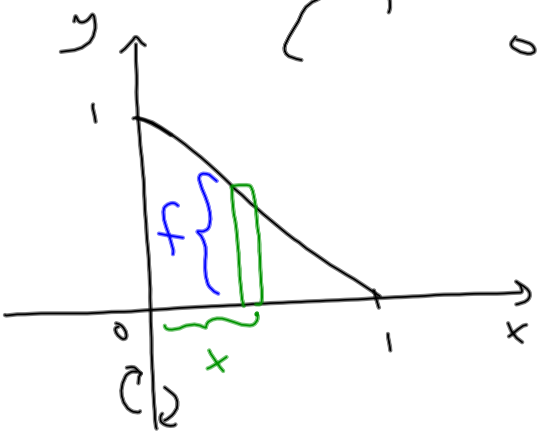
$$\boxed{V = 2\pi \int x f(x)\, dx}$$

Shell method

Eg: Find the volume of the solid obtained by revolving the region

$$\begin{cases} f(x) = 1-2x+3x^2-2x^3 \\ 0 \le x \le 1 \end{cases}$$

about the y-axis



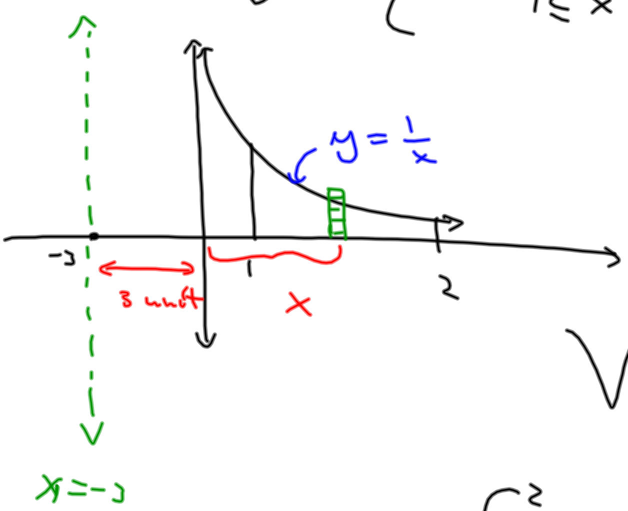
$$V = 2\pi\int_0^1 x\left(1-2x+3x^2-2x^3\right)dx$$

$$= 2\pi\left[\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{3}{4}x^4-\frac{2}{5}x^5\right]_0^1$$

$$\boxed{V = \frac{11\pi}{30}\ \text{unit}^3}$$

Ex:

Region $\begin{cases} y = \dfrac{1}{x} \\ 1 \le x \le 2 \end{cases}$ Rotate about $x = -3$



radius: $(x - (-3))$

height: $\dfrac{1}{x}$

$$\text{Volume: } 2\pi \int_1^2 (x+3)\left(\frac{1}{x}\right) dx$$

$$V = 2\pi \int_1^2 \left(1 + \frac{3}{x}\right) dx = 2\pi \Big[x + 3\ln x \Big]_1^2$$

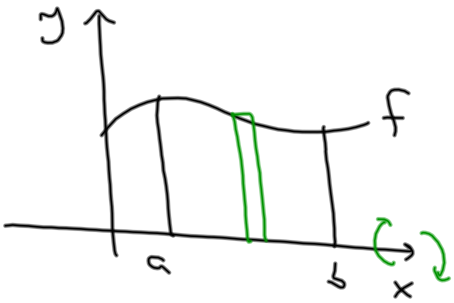
$$= 2\pi \Big[2 + 3\ln 2 - 1 + 3\underset{=\,\ln(1)}{(0)} \Big]$$

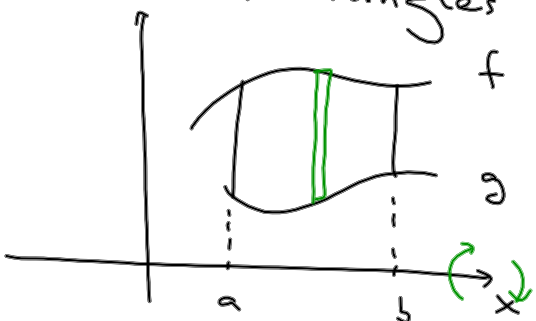
$$= 2\pi \Big[1 + 3\ln 2 \Big] \text{ unit}^3$$

$$\boxed{\simeq 19.35 \text{ unit}^3}$$

In Summary:

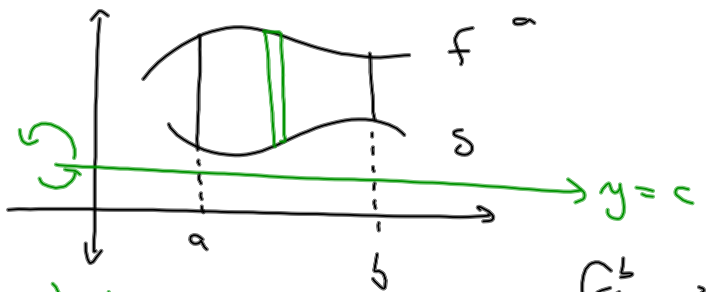
1. Disk or Washer method: perpendicular rectangles



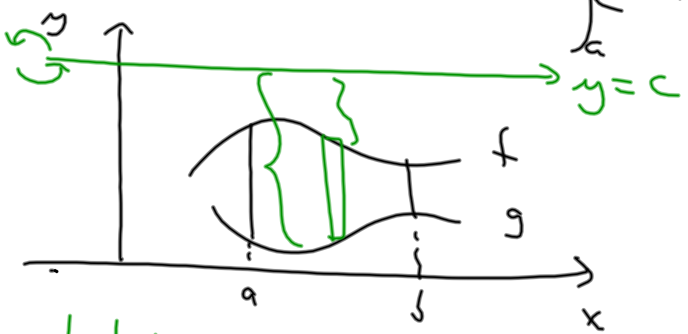


x-axis: Volume $= \pi \int_a^b f(x)^2 \, dx$

Volume: $\pi \int_a^b \left(\left(f(x)\right)^2 - \left(g(x)\right)^2 \right) dx$



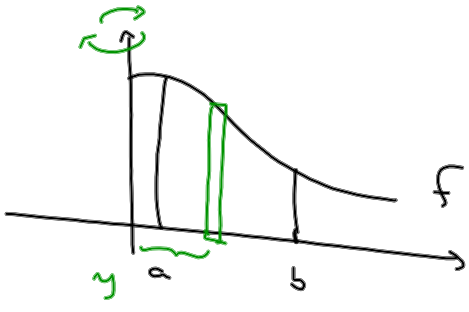
about $y = c$: Volume: $\pi \int_a^b \left[(f-c)^2 - (g-c)^2 \right] dx$

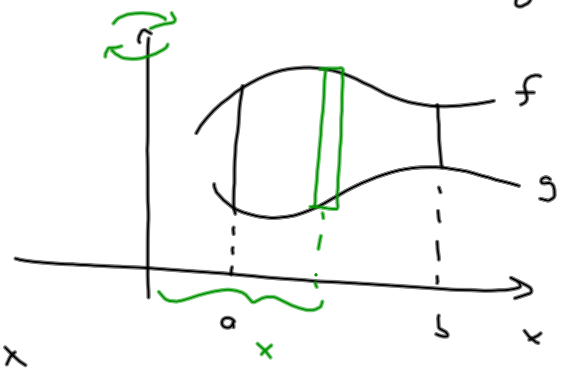


about $y = c$

Volume: $\pi \int_a^b \left[(c-g)^2 - (c-f)^2 \right] dx$

② Shell method : parallel rectangles

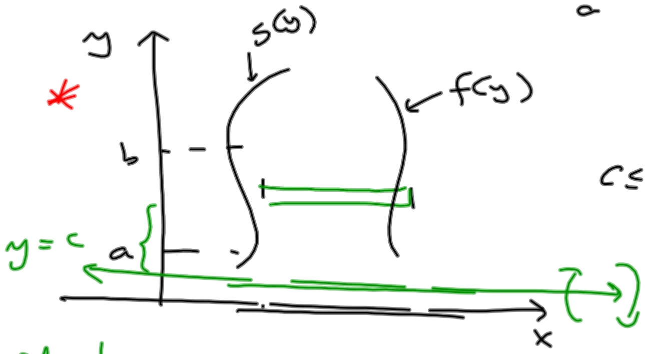




y-axis

Volume: $2\pi\int_a^b x\, f(x)\,dx$

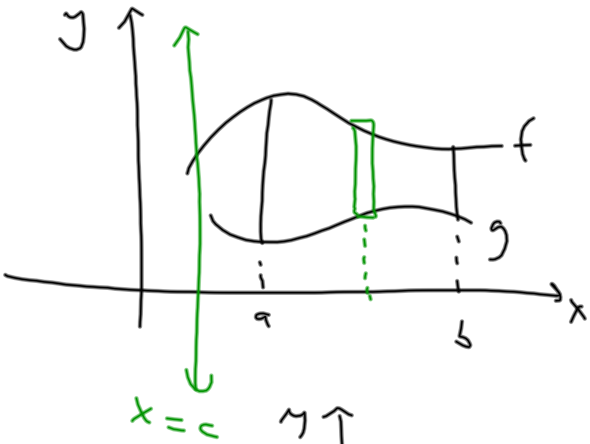
Volume: $2\pi\int_a^b x\left[f(x)-g(x)\right]dx$



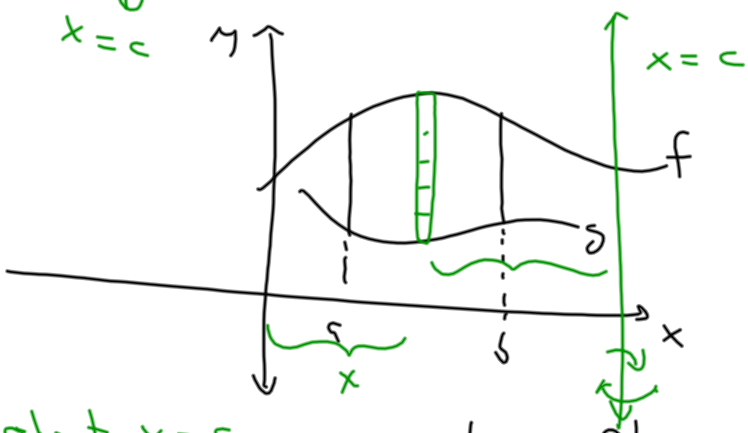
$c \le g \le f$

$$V = 2\pi\int_a^b (y-c)\big(f(y)-g(y)\big)\,dy$$

about y = c



$$V = 2\pi\int_a^b (x-c)\big(f(x)-g(x)\big)\,dx$$



about x = c

$$V = 2\pi\int_a^b (c-x)\big(f(x)-g(x)\big)\,dx$$